

# The fine structure of the stationary distribution for a simple Markov process\*

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## Abstract

We study the fractal properties of the stationary distribution  $\pi$  for a simple Markov process on  $\mathbb{R}$ . We will give bounds for the Hausdorff dimension of  $\pi$ , and lower bounds for the multifractal spectrum of  $\pi$ . Additionally, we will provide a method for numerically estimating these bounds.

## 1 Introduction

For real numbers  $\alpha > 1, \beta > 0$ , we define a Markov process by

$$X_{n+1} = \begin{cases} X_n + \beta & \text{with probability } p \\ \alpha^{-1} X_n & \text{with probability } 1 - p. \end{cases} \quad (1.1)$$

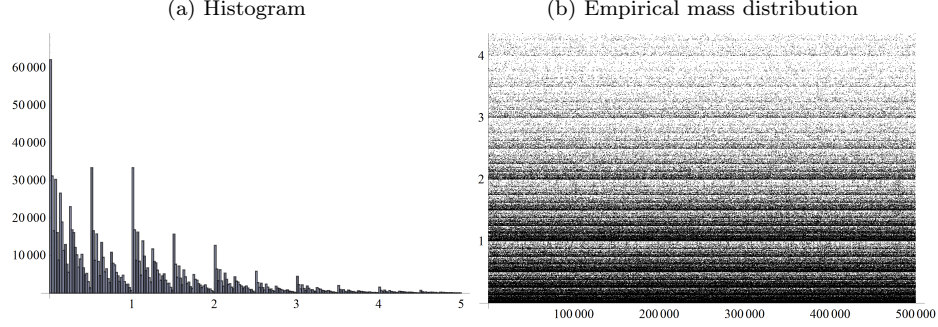
We will denote the stationary distribution of  $X_n$  by  $\pi$ . As figure 1.1 shows, this distribution exhibits typical fractal patterns. In order to acquire a solid framework in which we can study the fine structure (ie. Hausdorff dimension and multifractal spectrum) of  $\pi$ , we will reformulate the process  $X_n$  in the context of iterated function systems.

A (probabilistic) iterated function system (IFS) is a set  $\mathbb{X} \subset \mathbb{R}^d$  associated with a family of maps  $\mathcal{W} = \{w_i\}_{i=1}^N$ ,  $w_i : \mathbb{X} \rightarrow \mathbb{X}$ , where the maps are chosen independently according to a probability vector  $\mathbf{p} = \{p_i\}_{i \in M}$ , where  $p_i > 0$  for all  $i = 1, \dots, N$  and  $\sum_{i=1}^N p_i = 1$ . The maps are all Lipschitz, ie. there exist positive constants  $\gamma_i$  such that  $|w_i(x) - w_i(y)| \leq \gamma_i |x - y|$  for all  $x, y \in \mathbb{X}$  and  $i = 1, \dots, N$ . If  $\gamma_i < 1$  for all  $i$  the IFS is said to be strictly contracting, but a weaker condition is that  $\sum_{i=1}^N p_i \log \gamma_i < 0$ , in which case the IFS is said to fulfill

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Figure 1.1: Plot of  $X_n$  when  $\alpha = 2$ ,  $\beta = 1$  and  $p = 1/3$



average contractivity. In either case there exists a unique probability measure on  $\mathbb{X}$  satisfying

$$\mu = \sum_{i=1}^N p_i \mu \circ w_i^{-1},$$

which is called the invariant measure of the IFS (see [5] for a proof). In other terms, if we put  $\Sigma = \{1, 2, \dots, N\}$  and let  $\mathbb{P}$  be the infinite-fold product probability measure  $\mathbf{p} \times \mathbf{p} \times \dots$  on  $\Sigma^\infty$ , the limit

$$\nu(\mathbf{i}) = \lim_{n \rightarrow \infty} w_{i_1} \circ w_{i_2} \circ \dots \circ w_{i_n}(x_0) \quad (1.2)$$

exists for  $\mathbb{P}$ -almost every sequence  $\mathbf{i} = (i_1, i_2, \dots) \in \Sigma^\infty$ , and does not depend on  $x_0 \in \mathbb{X}$ . The mapping  $\nu : \Sigma^\infty \rightarrow \mathbb{X}$  is thus well defined almost everywhere on  $\mathbb{P}$  and  $\mu$  can be written as

$$\mu = \mathbb{P} \circ \nu^{-1}.$$

Now, let  $\Xi(\alpha, \beta, p)$  be the family of IFS's on  $\mathbb{R}$  of the form

$$\begin{aligned} w_1(x) &= x + \beta \\ w_2(x) &= \alpha^{-1}x \end{aligned}$$

with probability vector  $\mathbf{p} = (p, 1 - p)$ , and  $\alpha > 1, \beta > 0$ . The IFS isn't strictly contracting, since  $\gamma_1 = 1$ , but  $\sum p_i \log \gamma_i = -(1 - p) \log \alpha < 0$  shows that average contractivity holds. Thus the unique invariant measure  $\mu$  exists and satisfies the recursion relation

$$\mu(A) = p\mu(A - \beta) + (1 - p)\mu(\alpha A) \quad (1.3)$$

for any measurable  $A \subset \mathbb{R}$ . By writing  $X_{n+1} = w_{i_{n+1}}(X_n)$ , where  $i_n$  is drawn randomly according to  $\mathbb{P}$ , we see that the above IFS represents the same random process as the initial Markov process (1.1), and  $\mu$  is indeed equal to  $\pi$ . We will henceforth refer to this measure by  $\pi$ .

The following notions related to fractal geometry will largely follow the same definitions as in eg. [4]. The notation  $\dim_H$  will be used for the Hausdorff dimension of a set. For any Borel probability measure  $\mu$  on  $\mathbb{R}$ , the lower local dimension of  $\mu$  at  $x \in \mathbb{R}$  is defined by

$$\underline{\dim}\mu(x) = \liminf_{r \rightarrow 0} \frac{\log \mu(B(x, r))}{\log r} \quad (1.4)$$

The upper and lower Hausdorff dimensions of  $\mu$  are now given by

$$\dim_H^* \mu = \inf \{s : \underline{\dim}\mu(x) \leq s \text{ for } \mu\text{-almost all } x\} \quad (1.5)$$

$$\dim_H \mu = \sup \{s : \underline{\dim}\mu(x) \geq s \text{ for } \mu\text{-almost all } x\}, \quad (1.6)$$

respectively. Note that  $\dim_H \mu \leq \dim_H^* \mu$ . Now let

$$\overline{E}_t = \{x \in \mathbb{R} : \dim \pi(x) \leq t\}, \quad \underline{E}_t = \{x \in \mathbb{R} : \dim \pi(x) \geq t\},$$

and similarly  $\overline{f}_H(t) = \dim_H \overline{E}_t$ ,  $\underline{f}_H(t) = \dim_H \underline{E}_t$ . We call the functions  $\overline{f}_H(t)$  and  $\underline{f}_H(t)$  the upper and lower multifractal spectrum of  $\mu$ , respectively.

The Hausdorff dimension of invariant measures of IFS's have been studied extensively in the last decades. With light conditions on the maps in  $\mathcal{W}$  and only assuming average contractivity, in general only upper bounds for the Hausdorff dimension of  $\mu$  are known (see eg. [7]). The usual way of finding lower bounds is by trying to limit the overlap of the maps. This is most commonly done by assuming the open set condition (OSC), which is fulfilled if there exists an open set  $\mathcal{O} \subset \mathbb{X}$  such that  $w_i(\mathcal{O}) \subset \mathcal{O}$  and  $w_i(\mathcal{O}) \cap w_j(\mathcal{O}) = \emptyset$  for all  $i \neq j$ . If this condition fails, there are a few weaker assumptions that have yielded results (see [9] for a survey). In the simple case where the measure has compact support and the maps in  $\mathcal{W}$  are strictly contracting similitudes satisfying the OSC, the geometry is fully understood. The IFS we study here is of interest because it does not satisfy the OSC, nor any of the other overlap conditions. The only known result applicable to our process is

$$\dim_H^* \pi \leq \frac{p \log p + (1-p) \log(1-p)}{(1-p) \log \alpha^{-1}}.$$

In theorem 1.1 we present a strictly smaller upper bound, and a lower bound as well. We also obtain lower bounds for the multifractal spectrum.

For any positive integer  $b$  and  $x \in \mathbb{R}$ , let  $\delta_i^b(x)$  denote the  $i$ :th digit of a base- $b$  expansion of  $x/b^{\lfloor \log_b x \rfloor + 1} = 0.\delta_1^b \delta_2^b \dots \in [0, 1]$ . This representation is unique except for points whose expansion ends in an infinite sequence of 0's, since such numbers may also be written as an expansion ending in an infinite sequence of  $(b-1)$ 's. We will ensure uniqueness of  $\delta_i^b(x)$  by always choosing the former representation in such cases. Write  $\tau_k^b(x, n)$  for the number of occurrences of the digit  $k$  in the  $n$  first digits of the base- $b$  expansion of  $x$ . Whenever it exists, we denote  $\tau_k^b(x) = \lim_{n \rightarrow \infty} \frac{1}{n} \tau_k^b(x, n)$ . For any vector  $(q_0, q_1, \dots, q_{b-1})$  of non-negative real numbers with  $\sum_{i=0}^{b-1} q_i = 1$  we define

$$F_b(q_0, q_1, \dots, q_{b-1}) = \{x \in \mathbb{R} : \tau_k^b(x) = q_k, k = 0, 1, \dots, b-1\}.$$

Furthermore, let

$$S_{b,t} = \left\{ x \in \mathbb{R} : \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \delta_i^b(x) = t \right\} = \left\{ x \in \mathbb{R} : \sum_{i=0}^{b-1} i \tau_i^b(x) = t \right\}.$$

By a classical result of Eggleston ([2]),

$$\dim_H (F_b(q_0, q_1, \dots, q_{b-1})) = - \sum_{i=1}^{b-1} q_i \log q_i / \log b. \quad (1.7)$$

The Hausdorff dimension of the set  $S_{b,t}$  is known to be (see [1], corollary 15)

$$\dim_H (S_{b,t}) = \frac{\log \min \left\{ \sum_{i=0}^{b-1} r^{i-t} : r > 0, \sum_{i=0}^{b-1} (i-t)r^i = 0 \right\}}{\log b}. \quad (1.8)$$

For any  $\alpha > 1$  and  $0 \leq p \leq 1$ , we set

$$\bar{d}(x) = \frac{\log(1-p) + x \log p}{\log \alpha^{-1}}, \quad \underline{d}(x) = \bar{d}(x) - \frac{\log(1-p^{\alpha-1})}{\log \alpha^{-1}}$$

and

$$d^*(x) = \min \left\{ \max \left\{ \bar{d}^{-1}(x), 0 \right\}, 1 \right\}, \quad d_*(x) = \min \left\{ \max \left\{ \underline{d}^{-1}(x), 0 \right\}, 1 \right\}$$

for all  $0 \leq x \leq 1$ . We are now ready to state the main result:

**Theorem 1.1.** *For an IFS in  $\Xi(\alpha, \beta, p)$  where  $\alpha \geq 2$  and  $\beta \geq 1$  are integers we have*

$$\dim_H^* \pi \leq \frac{\sum_{i=0}^{\alpha-1} \xi_i \log \xi_i}{\log \alpha^{-1}}, \quad (1.9)$$

where  $\xi_k = \sum_{m=0}^{\infty} \pi[m\alpha + k, m\alpha + k + 1]$ . Moreover, if  $p \leq 1/2$  and  $\beta = \alpha^t$  for some  $t = 0, 1, 2, \dots$ , then

$$\dim_H \pi \geq \underline{d} \left( \sum_{i=0}^{\alpha-1} i \xi_i \right) \quad (1.10)$$

and

$$\bar{f}_H(t) \geq \dim_H (S_{\alpha, d^*(t)}), \quad \underline{f}_H(t) \geq \dim_H (S_{\alpha, d_*(t)}). \quad (1.11)$$

## 2 Statement of results

In this section, if not otherwise stated, we will assume that  $\pi$  is the invariant measure of an IFS in  $\Xi(\alpha, \beta, p)$ , where  $p \leq 1/2$  and  $\alpha \geq 2$  and  $\beta \geq 1$  are integers.

**Lemma 2.1.** *For any non-negative  $x$  we have*

$$\pi[x, \alpha x] = \frac{p}{1-p} \pi[x - \beta, x].$$

*Proof.*

$$\begin{aligned}
 \pi[x, \alpha x] &= \pi[0, \alpha x] - \pi[0, x] \\
 &= \pi[0, \alpha x] - (1-p)\pi[0, \alpha x] - p\pi[0, x - \beta] \\
 &= p(\pi[x - \beta, x] + \pi[x, \alpha x])
 \end{aligned}$$

□

**Lemma 2.2.** Write  $M_0 = \pi[0, \beta]$ . There exists a constant  $K > 1$  such that

$$M_0 p^n \leq \pi[n\beta, (n+1)\beta] \leq M_0 K p^n$$

for all integers  $n \geq 0$ .

*Proof.* Assume that  $n \geq 1$  (If  $n = 0$ , the proposition holds for any  $K \geq 1$ ). The lower bound follows immediately from (1.3) since

$$\pi[n\beta, (n+1)\beta] \geq p\pi[(n-1)\beta, n\beta] \geq p^2\pi[(n-2)\beta, (n-1)\beta] \geq \dots \geq p^n M_0$$

For the upper bound, we first use lemma 2.1 and the facts that  $\alpha \geq 2$  and  $p \leq 1/2$  to note that

$$\pi[n\beta, (n+1)\beta] \leq \pi[n\beta, n\alpha\beta] = \frac{p}{1-p}\pi[(n-1)\beta, n\beta] \leq \pi[(n-1)\beta, n\beta].$$

This implies that  $\pi[m\beta, n\beta] \geq (n-m)\pi[(n-1)\beta, n\beta]$  for any integers  $n > m \geq 0$ . Thus

$$\frac{p}{1-p}\pi[(n-1)\beta, n\beta] = \pi[n\beta, n\alpha\beta] \geq n(\alpha-1)\pi[(n\alpha-1)\beta, n\alpha\beta].$$

The above and lemma 2.1 give

$$\begin{aligned}
 \pi[n\beta, (n+1)\beta] &\leq p\pi[(n-1)\beta, n\beta] + (1-p)\pi[n\alpha\beta, n\alpha^2\beta] \\
 &= p(\pi[(n-1)\beta, n\beta] + p\pi[(n\alpha-1)\beta, n\alpha\beta]) \quad (2.1)
 \end{aligned}$$

$$\leq p\pi[(n-1)\beta, n\beta] \left(1 + \frac{p}{1-p} \cdot \frac{1}{n(\alpha-1)}\right). \quad (2.2)$$

For any integers  $n \geq m \geq 1$ , let  $P(x, m, n) = \prod_{k=n-m+1}^n (1 + x/k)$ . By writing  $a = \frac{p}{(1-p)(\alpha-1)}$  and repeating (2.2) we get

$$\pi[n\beta, (n+1)\beta] \leq p^m \pi[(n-m)\beta, (n-m+1)\beta] P(a, m, n) \quad (2.3)$$

Now, apply (2.3) to the second term in (2.1) to see that

$$\pi[n\beta, (n+1)\beta] \leq p\pi[(n-1)\beta, n\beta] \left(1 + p^{n(\alpha-1)} P(a, n(\alpha-1), n\alpha-1)\right)$$

A standard result is that  $P(x, n, n) = \Gamma(x+n)/\Gamma(x)$  (where  $\Gamma(x)$  denotes the gamma function), which implies  $P(1, n, n) = n+1$ . Since  $a \leq 1$  we have  $P(a, n(\alpha-1), n\alpha-1) < P(1, n, n\alpha-1) = n\alpha$ . Thus we arrive at

$$\pi[n\beta, (n+1)\beta] < p^n M_0 \prod_{k=1}^n \left(1 + k\alpha p^{k(\alpha-1)}\right) < p^n M_0 \prod_{k=1}^{\infty} \left(1 + k\alpha p^{k(\alpha-1)}\right).$$

The infinite product above converges if and only if the series  $\sum_{k=1}^{\infty} k\alpha p^{k(\alpha-1)}$  converges, which is clearly the case.  $\square$

**Lemma 2.3.** *For any integers  $n$  and  $k$ ,*

$$\pi \left[ \frac{n\beta}{\alpha^k}, \frac{(n+1)\beta}{\alpha^k} \right] = (1-p) \sum_{k=0}^{\lfloor n\alpha^{-k} \rfloor} p^k \pi \left[ \frac{n\beta}{\alpha^{k-1}} - k\alpha\beta, \frac{(n+1)\beta}{\alpha^{k-1}} - k\alpha\beta \right].$$

*Proof.* The formula is straightforward to obtain using (1.3). We have

$$\begin{aligned} & \pi \left[ \frac{n\beta}{\alpha^k}, \frac{(n+1)\beta}{\alpha^k} \right] \\ &= p\pi \left[ \frac{n\beta}{\alpha^k} - \beta, \frac{(n+1)\beta}{\alpha^k} - \beta \right] + (1-p)\pi \left[ \frac{n\beta}{\alpha^{k-1}}, \frac{(n+1)\beta}{\alpha^{k-1}} \right]. \end{aligned}$$

The first term above can be written as

$$\begin{aligned} p\pi \left[ \frac{n\beta}{\alpha^k} - \beta, \frac{(n+1)\beta}{\alpha^k} - \beta \right] &= p^2\pi \left[ \frac{n\beta}{\alpha^k} - 2\beta, \frac{(n+1)\beta}{\alpha^k} - 2\beta \right] + \\ & p(1-p)\pi \left[ \frac{n\beta}{\alpha^{k-1}} - \alpha\beta, \frac{(n+1)\beta}{\alpha^{k-1}} - \alpha\beta \right]. \end{aligned}$$

By repeatedly using (1.3) on the first terms, we generally have

$$\begin{aligned} & p^j\pi \left[ \frac{n\beta}{\alpha^k} - j\beta, \frac{(n+1)\beta}{\alpha^k} - j\beta \right] = \\ & p^{j+1}\pi \left[ \frac{n\beta}{\alpha^k} - (j+1)\beta, \frac{(n+1)\beta}{\alpha^k} - (j+1)\beta \right] \\ & + p^j(1-p)\pi \left[ \frac{n\beta}{\alpha^{k-1}} - j\alpha\beta, \frac{(n+1)\beta}{\alpha^{k-1}} - j\alpha\beta \right]. \end{aligned}$$

Combining everything yields

$$\begin{aligned} \pi \left[ \frac{n\beta}{\alpha^k}, \frac{(n+1)\beta}{\alpha^k} \right] &= p^{\lfloor n\alpha^{-k} \rfloor} \pi \left[ \beta \left( \frac{n}{\alpha^k} - \left\lfloor \frac{n}{\alpha^k} \right\rfloor \right), \beta \left( \frac{n+1}{\alpha^k} - \left\lfloor \frac{n}{\alpha^k} \right\rfloor \right) \right] + \\ & (1-p) \sum_{j=0}^{\lfloor n\alpha^{-k} \rfloor - 1} p^j \pi \left[ \frac{n\beta}{\alpha^{k-1}} - j\alpha\beta, \frac{(n+1)\beta}{\alpha^{k-1}} - j\alpha\beta \right]. \end{aligned}$$

Since  $\alpha$  is an integer, we have  $\frac{n}{\alpha^k} - \left\lfloor \frac{n}{\alpha^k} \right\rfloor \geq 0$  and  $\frac{n+1}{\alpha^k} - \left\lfloor \frac{n}{\alpha^k} \right\rfloor \leq 1$  for all  $n, k$ . Thus

$$\begin{aligned} & \pi \left[ \beta \left( \frac{n}{\alpha^k} - \left\lfloor \frac{n}{\alpha^k} \right\rfloor \right), \beta \left( \frac{n+1}{\alpha^k} - \left\lfloor \frac{n}{\alpha^k} \right\rfloor \right) \right] \\ &= (1-p)\pi \left[ \frac{n\beta}{\alpha^{k-1}} - \left\lfloor \frac{n}{\alpha^k} \right\rfloor \alpha\beta, \frac{(n+1)\beta}{\alpha^{k-1}} - \left\lfloor \frac{n}{\alpha^k} \right\rfloor \alpha\beta \right] \end{aligned}$$

whereby the proposition follows.  $\square$

**Lemma 2.4.** *For all integers  $n, k \geq 0$ , define*

$$g(n, k) = n\alpha + (1 - \alpha) \sum_{i=0}^k \lfloor n\alpha^{-i} \rfloor.$$

*Let  $n \geq 0$  be arbitrary. Then, for all  $k \geq 0$ ,*

$$M_0(1-p)^k p^{g(n,k)} \leq \pi \left[ \frac{n\beta}{\alpha^k}, \frac{(n+1)\beta}{\alpha^k} \right] \leq M_0 K \left( \frac{1-p}{1-p^{\alpha-1}} \right)^k p^{g(n,k)}.$$

*Proof.* The proposition holds for  $k = 0$  by lemma 2.2. Assume that it holds for  $k = t - 1$ , for some  $t > 1$ . Then, by lemma 2.3,

$$\begin{aligned} \pi \left[ \frac{n\beta}{\alpha^t}, \frac{(n+1)\beta}{\alpha^t} \right] &= (1-p) \sum_{j=0}^{\lfloor n\alpha^{-t} \rfloor} p^j \pi \left[ \frac{(n-j\alpha^t)\beta}{\alpha^{t-1}}, \frac{(n+1-j\alpha^t)\beta}{\alpha^{t-1}} \right] \\ &\leq M_0 K \frac{(1-p)^t}{(1-p^{\alpha-1})^{t-1}} \sum_{j=0}^{\lfloor n\alpha^{-t} \rfloor} p^j \cdot p^{g(n-j\alpha^t, t-1)}. \end{aligned}$$

Notice that

$$\begin{aligned} g(n-j\alpha^t, t-1) &= (n-j\alpha^t)\alpha + (1-\alpha) \sum_{m=0}^{t-1} \left\lfloor \frac{n-j\alpha^t}{\alpha^m} \right\rfloor \\ &= g(n, t-1) + j \left( -\alpha^{t+1} - (1-\alpha) \sum_{m=0}^{t-1} \alpha^{t-m} \right) \\ &= g(n, t-1) - j\alpha. \end{aligned}$$

Thus

$$\pi \left[ \frac{n\beta}{\alpha^t}, \frac{(n+1)\beta}{\alpha^t} \right] \leq M_0 K \frac{(1-p)^t}{(1-p^{\alpha-1})^{t-1}} p^{g(n, t-1)} \sum_{j=0}^{\lfloor n\alpha^{-t} \rfloor} p^{j(1-\alpha)}$$

Now, since

$$p^{(1-\alpha)\lfloor n\alpha^{-t} \rfloor} \leq \sum_{j=0}^{\lfloor n\alpha^{-t} \rfloor} p^{j(1-\alpha)} \leq \frac{p^{(1-\alpha)\lfloor n\alpha^{-t} \rfloor}}{1-p^{\alpha-1}}, \quad (2.4)$$

and  $g(n, t-1) + (1-\alpha)\lfloor n\alpha^{-t} \rfloor = g(n, t)$ , we have

$$\pi \left[ \frac{n\beta}{\alpha^t}, \frac{(n+1)\beta}{\alpha^t} \right] \leq M_0 K \left( \frac{1-p}{1-p^{\alpha-1}} \right)^t p^{g(n, t)}$$

For the lower bound, we use a practically identical calculation and the lower bound in (2.4) to obtain

$$\begin{aligned} \pi \left[ \frac{n\beta}{\alpha^t}, \frac{(n+1)\beta}{\alpha^t} \right] &\geq M_0(1-p)^t p^{g(n, t-1)} \sum_{j=0}^{\lfloor n\alpha^{-t} \rfloor} p^{j(1-\alpha)} \\ &\geq M_0(1-p)^t p^{g(n, t)}. \end{aligned}$$

□

**Lemma 2.5.** *Let  $x \in D_\alpha$ . For any integer  $b \geq 2$ , define*

$$\underline{\sigma}^b(x) = \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \delta_i^b(x).$$

*Then*

$$\underline{d}(\underline{\sigma}^\alpha(x/\beta)) \leq \underline{\dim} \pi(x) \leq \bar{d}(\underline{\sigma}^\alpha(x/\beta)).$$

*Proof.* First, we remark that for any integer  $n \geq 0$  the quantity  $g(n, k)$  is related to the sum of the digits in the base- $\alpha$  expansion of  $n$ . Define  $L(x) = \lfloor \log_\alpha x \rfloor + 1$ , then

$$\begin{aligned} g(n, k) &= \sum_{j=0}^k \lfloor n\alpha^{-j} \rfloor - \alpha \left( \sum_{j=0}^k \lfloor n\alpha^{-j} \rfloor - n \right) \\ &= \sum_{j=0}^k \lfloor n\alpha^{-j} \rfloor - \alpha \sum_{j=0}^{k-1} \lfloor n\alpha^{-j-1} \rfloor \\ &= \lfloor n\alpha^{-k} \rfloor + \sum_{i=0}^{k-1} \delta_{L(n)-i}^\alpha(n), \end{aligned}$$

since  $\lfloor n\alpha^{-j} \rfloor - \alpha \lfloor n\alpha^{-j-1} \rfloor = \delta_{L(n)-j}^\alpha(n)$ . Now, fix  $x \in \mathbb{R}$  and take  $\{x_k\}_{k=0}^\infty$  to be the unique sequence of integers satisfying

$$x \in \left[ \frac{x_k \beta}{\alpha^k}, \frac{(x_k + 1) \beta}{\alpha^k} \right) \quad (2.5)$$

for every  $k \geq 0$ . Additionally, fix  $r \in (0, \beta\alpha^{-1}]$  and put  $k = \max \{k \in \mathbb{N} : r \leq \beta\alpha^{-k}\}$ . Then  $r > \beta\alpha^{-k-1}$  and

$$\frac{\log \pi(B(x, r))}{\log r} \geq \frac{\log \pi[(x_k - 1)\beta\alpha^{-k}, (x_k + 2)\beta\alpha^{-k}]}{\log \beta\alpha^{-k-1}}.$$

Define  $\bar{x}_k$  as the integer in  $\{x_k - 1, x_k, x_k + 1\}$  for which  $\pi[\bar{x}_k\beta\alpha^{-k}, (\bar{x}_k + 1)\beta\alpha^{-k}]$  is maximized. Then

$$\begin{aligned} \frac{\log \pi(B(x, r))}{\log r} &\geq \frac{\log 3\pi[\bar{x}_k\beta\alpha^{-k}, (\bar{x}_k + 1)\beta\alpha^{-k}]}{\log \beta\alpha^{-k-1}} \\ &\geq \frac{\log 3M_0K + \log \left( \frac{1-p}{1-p^{\alpha-1}} \right)^k p^{g(\bar{x}_k, k)}}{(k+1) \log \alpha^{-1} + \log \beta}, \end{aligned}$$

where we applied lemma (2.4) in the second step. Now set  $y_k = \bar{x}_k\beta\alpha^{-k}$  and  $N = \min \{n : \beta < \alpha^n\}$ . By (2.5),

$$|x - y_k| < 2\beta\alpha^{-k} \leq \alpha^{N+1-k}, \quad (2.6)$$



for  $k \geq 0$ , implying  $\delta_i(x) = \delta_i(y_k)$  for  $1 \leq i \leq k'$  where  $k' = k - N - 1$ . Thus

$$\sum_{i=1}^{k'} \delta_i(\bar{x}_k) = \sum_{i=1}^{k'} \delta_i\left(\frac{\alpha^k}{\beta} y_k\right) = \sum_{i=1}^{k'} \delta_i\left(\frac{\alpha^k}{\beta} x\right)$$

for all  $k \geq 0$ , giving

$$\frac{g(\bar{n}_k, k')}{k} = \frac{1}{k} \left( \left\lfloor \bar{x}_k \alpha^{-k'} \right\rfloor + \sum_{i=1}^{k'} \delta_i\left(\frac{\alpha^k}{\beta} x\right) \right). \quad (2.7)$$

Multiplying a number by  $\alpha^k$  does not affect its digits, so as  $r \rightarrow 0$ ,

$$\begin{aligned} \liminf_{r \rightarrow 0} \frac{\log \pi(B(x, r))}{\log r} &\geq \liminf_{k \rightarrow \infty} \frac{\log 3M_0K + k \log \left( \frac{1-p}{1-p^{\alpha-1}} \right) + g(\bar{x}_k, k) \log p}{(k+1) \log \alpha^{-1} + \log \beta} \\ &= \frac{\log(1-p) - \log(1-p^{\alpha-1}) + \underline{\sigma}(x/\beta) \log p}{\log \alpha^{-1}}, \end{aligned}$$

since  $k'/k \rightarrow 1$ . For the upper bound, fix  $r$  and  $k$  as before, then

$$\begin{aligned} \frac{\log \pi(B(x, r))}{\log r} &\leq \frac{\log \pi[x_{k+1}\beta\alpha^{-k-1}, (x_{k+1}+1)\beta\alpha^{-k-1}]}{\log \beta\alpha^{-k}} \\ &\leq \frac{\log(M_0(1-p)^{k+1}p^{g(x_{k+1}, k+1)})}{k \log \alpha^{-1} + \log \beta}. \end{aligned}$$

As  $r$  decreases and  $k$  increases,  $|x - x_{k+1}\beta\alpha^{-k}| < \beta\alpha^{-k}$  so (2.7) holds, whereby

$$\begin{aligned} \liminf_{r \rightarrow 0} \frac{\log \pi(B(x, r))}{\log r} &\leq \liminf_{k \rightarrow \infty} \frac{\log M_0 + (k+1) \log(1-p) + g(x_{k+1}, k+1) \log p}{k \log \alpha^{-1} + \log \beta} \\ &= \frac{\log(1-p) + \underline{\sigma}(x/\beta) \log p}{\log \alpha^{-1}}. \end{aligned}$$

□

**Lemma 2.6.** *For  $\pi$ -almost every  $x$  we have*

$$\tau_k^\alpha(x) = \xi_k$$

for  $k = 0, 1, \dots, \alpha - 1$ .

*Proof.* Let  $X_n$  be as in (1.1), and write  $X_n(\mathbf{i}) = w_{i_n} \circ w_{i_{n-1}} \cdots \circ w_{i_1}(X_0)$  for  $\mathbf{i} \in \Sigma^\infty$ . Define  $n_\alpha(X_n)$  as the number of digits in the base- $\alpha$  expansion of  $X_n - \lfloor X_n \rfloor$ , and  $n'_\alpha(X_n)$  as the number of digits in  $\lfloor X_n \rfloor$ . The number  $n_\alpha(X_n)$  will equal the number of times the map  $w_2$  is chosen, so for  $\mathbb{P}$ -almost every  $\mathbf{i}$ ,

$$\lim_{n \rightarrow \infty} \frac{n_\alpha(X_n(\mathbf{i}))}{n} = 1 - p, \quad (2.8)$$

by the law of large numbers. On the other hand,  $\lfloor X_n(\mathbf{i}) \rfloor$  is at most equal to the number of times  $w_1$  is chosen, so

$$\limsup_{n \rightarrow \infty} \frac{\lfloor X_n(\mathbf{i}) \rfloor}{n} \leq p,$$

$\mathbb{P}$ -a.e. It follows that

$$\limsup_{n \rightarrow \infty} \frac{n'_\alpha(X_n(\mathbf{i}))}{n_\alpha(X_n(\mathbf{i}))} \leq \lim_{n \rightarrow \infty} \frac{\lfloor \log_\alpha np \rfloor + 1}{n(1-p)} = 0$$

$\mathbb{P}$ -a.e., which shows that the integer part does not contribute to the asymptotical frequency of digits, i.e. it suffices to analyze  $\tau_k^\alpha(X_n - \lfloor X_n \rfloor)$ .

Let  $Y_n = (X_n, i_{n+1})$  and observe that  $Y_n$  is a Markov chain with state space  $\mathbb{X} = [0, \infty) \times \{1, 2\}$  and stationary distribution  $\pi_Y = \pi \times \mathbb{P}$ . Set  $A = \{(x, 2) : x \in [0, \infty)\}$  and let  $T^n(A)$  denote the  $n$ :th visit of  $Y_n$  in  $A$ . Since  $Y_n$  is ergodic,  $Z_n = Y_{T^n(A)}$  is also a Markov chain, with stationary distribution  $\pi_Z = \pi_Y / \pi_Y(A) = \pi_Y / (1-p)$ . Now define  $h_k : \mathbb{X} \rightarrow \{0, 1\}$  by

$$h_k(Y_n) = \begin{cases} 1, & \text{if } \lfloor X_n \rfloor \bmod \alpha = k \text{ and } i_{n+1} = 2 \\ 0, & \text{otherwise.} \end{cases}$$

Informally, whenever  $X_{n+1}$  adds a digit to the  $\alpha$ -expansion of  $X_n - \lfloor X_n \rfloor$ ,  $h_k(Y_{n+1})$  will equal 1 if the added digit is  $k$ . This means that

$$\tau_k^\alpha(Z_{1,n} - \lfloor Z_{1,n} \rfloor, n) = \sum_{i=1}^n h_k(Z_i),$$

where  $Z_{1,n}$  denotes the first coordinate of  $Z_n$ . While  $h_k$  is not continuous on  $\mathbb{X}$ , it is continuous on  $([0, \infty) \setminus \mathbb{Z}) \times \{1, 2\}$ . Thus, for any  $\epsilon > 0$ , we can find continuous functions  $\bar{h}_{k,\epsilon}, \underline{h}_{k,\epsilon} : \mathbb{X} \rightarrow [0, 1]$  such that  $\underline{h}_{k,\epsilon}(Y_n) < h_k(Y_n) < \bar{h}_{k,\epsilon}(Y_n)$  for all  $n \geq 0$  and for any  $m = 0, 1, 2, \dots$ ,

$$\begin{aligned} \bar{h}_{k,\epsilon}(x, 2) &= \begin{cases} \leq 1, & \text{for any } x \in [m\alpha + k - \epsilon, m\alpha + k + 1 + \epsilon) \\ 0, & \text{otherwise} \end{cases} \\ \underline{h}_{k,\epsilon}(x, 2) &= \begin{cases} 1, & \text{for any } x \in [m\alpha + k - \epsilon, m\alpha + k + 1 + \epsilon) \\ < 1, & \text{otherwise} \end{cases} \end{aligned}$$

Now, by an ergodic theorem of Elton ([3]), for  $f = \bar{h}_{k,\epsilon}, \underline{h}_{k,\epsilon}$ , for  $\mathbb{P}$ -almost every  $\mathbf{i}$ ,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^{n-1} f(Z_i(\mathbf{i})) = \int f d\pi_Z, \quad (2.9)$$

for all initial points  $Z_0 \in \mathbb{X}$ . Thus, for every  $\epsilon > 0$ ,

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{1}{n} \tau_k^\alpha(Z_{1,n}, n) &< \sum_{m=0}^{\infty} \pi[m\alpha + k - \epsilon, m\alpha + k + 1 + \epsilon] \\ \liminf_{n \rightarrow \infty} \frac{1}{n} \tau_k^\alpha(Z_{1,n}, n) &> \sum_{m=0}^{\infty} \pi[m\alpha + k + \epsilon, m\alpha + k + 1 - \epsilon]. \end{aligned}$$

This means that for  $k = 0, 1, \dots, \alpha - 1$ ,  $\mathbb{P}$ -a.e.,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \tau_k^\alpha(Z_{1,n}, n) = \xi_k \quad (2.10)$$

where  $Z_{1,n}(\mathbf{i}) = X_{T^n(A)}(\mathbf{i})$ . The convergence is independent of  $X_0$ . Now define the “backward” process  $\tilde{X}_n(\mathbf{i}) = w_{i_1} \circ w_{i_2} \circ \dots \circ w_{i_n}(X_0)$ . By (1.2),  $\tilde{X}_n$  converges  $\mathbb{P}$ -a.e. to  $\nu(\mathbf{i})$ , which has distribution  $\pi$ , since the distribution of  $X_n$  (which is the same for  $\tilde{X}_n$ ) converges to  $\pi$ . Furthermore, (2.8) must also hold for  $\tilde{X}_n$  since  $\tilde{X}_n$  has the same distribution as  $X_n$ . As  $n_\alpha(X_n(\mathbf{i})) \rightarrow \infty$ , (2.10) implies

$$\frac{\tau_k^\alpha(X_n(\mathbf{i}), n_\alpha(X_n(\mathbf{i})))}{n_\alpha(X_n(\mathbf{i}))} \rightarrow \xi_k,$$

$\mathbb{P}$ -a.e., and the same claim must again also hold for  $\tilde{X}_n$ . It follows that  $\mathbb{P}$ -a.e.,  $\tau_k^\alpha(\nu(\mathbf{i}), n)/n \rightarrow \xi_k$ , and the proof is complete.  $\square$

Our main theorem now follows from the above lemmas:

*Proof of theorem 1.1.* Lemma 2.6 implies that  $\pi(F_\alpha(\xi_0, \xi_1, \dots, \xi_{\alpha-1})) = 1$ , so (1.9) follows immediately from (1.7). Now, assume that  $\beta = \alpha^t$  for some  $t = 0, 1, \dots$ . Then, for any  $x$ ,  $x/\beta$  will have the same digit expansion as  $x$ . Thus, lemmas 2.6 and 2.5 together give (1.10). For the last part, note that for any  $x \in S_{\alpha, \bar{d}^{-1}(t)}$ , lemma 2.5 implies  $\dim \pi(x) \leq t$  and thus  $x \in \bar{E}_t$ . An analogous argument shows that  $x \in S_{\alpha, \underline{d}^{-1}(t)}$  implies  $x \in \underline{E}_t$ , whereby (1.11) follows.  $\square$

*Remark.* If we replace  $\liminf$  by  $\limsup$  in (1.4) and (1.5)-(1.6), we obtain the definitions of the upper and local *packing* dimensions of  $\mu$ , denoted  $\dim_P \mu$  and  $\dim_P^* \mu$ , respectively. If  $x \in S_{\alpha, y}$  for any  $y \in [0, 1]$ , the limit inferior in lemma 2.5 may be dropped in favor of the ordinary limit. Thus, in the (latter) setting of theorem 1.1,

$$\underline{d} \left( \sum_{i=1}^{\alpha-1} i \xi_i \right) \leq \dim_P \pi \leq \dim_P^* \pi \leq \bar{d} \left( \sum_{i=1}^{\alpha-1} i \xi_i \right).$$

### 3 Numerical estimates

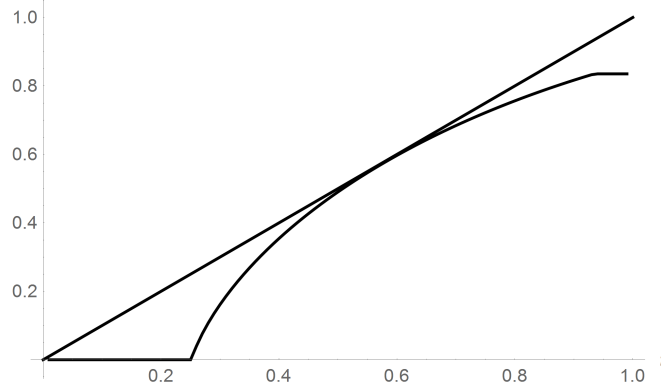
When  $\beta = 1$  we can use the following method to find numerical approximations of the dimension values in theorem 1.1. Since we only need to evaluate the  $\pi$ -mass of intervals of unit length, we partition the state space of  $X_n$  according to  $[0, \infty) = \bigcup_{i=1}^{\infty} A_i$  where  $A_i = [i-1, i)$ . Now we define a new process  $X'_n$  on  $\mathbb{N}$  by the transition probabilities

$$X'_{n+1} = \begin{cases} X_n + 1, & \text{with probability } p \\ \lfloor X_n \alpha^{-1} \rfloor, & \text{with probability } 1 - p \end{cases}$$

Note that  $X'_n = m$  whenever  $X_n \in A_m$ , since  $\alpha$  is an integer and

$$[m\alpha^{-1}, (m+1)\alpha^{-1}] \subset [\lfloor m\alpha^{-1} \rfloor, \lfloor m\alpha^{-1} \rfloor + 1]$$

Figure 3.1: Lower and upper bounds for  $\bar{f}_H(t)$ , when  $\alpha = 5$ ,  $p = 1/3$ .



for all  $m \in \mathbb{N}$  (as in lemma 2.3). The process  $X'_n$  is called a lumped process of  $X_n$  (see [8], section 6.3). Clearly,  $X'_n$  is a Markov process itself, and it is easily seen that it has stationary distribution  $\pi'$  defined by  $\pi'(m) = \pi(A_m)$  for all  $m \in \mathbb{N}$ .

We now define the truncated matrix

$$P_n(i, j) = \begin{cases} p, & i = j = n \\ P(i, j), & \text{otherwise} \end{cases}$$

where the “missing” probability is added to the last state to ensure that the matrix remains stochastic. If we consider the finite system  $\pi_n = \pi_n P_n$ , by results of Heyman ([6]),

$$\lim_{n \rightarrow \infty} \pi_n(m) = \pi'(m)$$

for all  $m \in \mathbb{N}$ . This implies that  $\lim_{n \rightarrow \infty} \sum_{i=0}^{\infty} \pi_n(k + i\alpha + 1) = \xi_k$ , so by calculating the left eigenvectors of  $P_n$  for some large value of  $n$  we can find estimates for the dimension of  $\pi$  using theorem 1.1. For example, if  $\alpha = 2$  and  $\beta = 1$  we have

$$P_5 = \begin{pmatrix} 1-p & p & 0 & 0 & 0 \\ 1-p & 0 & p & 0 & 0 \\ 0 & 1-p & 0 & p & 0 \\ 0 & 1-p & 0 & 0 & p \\ 0 & 0 & 1-p & 0 & p \end{pmatrix}.$$

Let  $p = 1/3$ . Now, by calculating the left eigenvectors of  $P_{50}$ , we have

$$0.508 \leq \dim_H \pi \leq \dim_H^* \pi \leq 0.906.$$

The bounds are tighter for larger values of  $\alpha$ . If we take  $\alpha = 5$  instead, we have

$$0.579 \leq \dim_H \pi \leq \dim_H^* \pi \leq 0.585.$$

In this case, the lower bound to  $\bar{f}_H(t)$  given by theorem 1.1, along with the upper bound  $\bar{f}_H(t) \leq t$  (this is standard, see eg. [4]) are plotted in figure (3.1). Note that these bounds hold for every  $\beta = \alpha^k$ , where  $k \geq 0$  is an integer.

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